VORTICES, FLUXES AND UNIVERSALITY
&
EXPLORING CENTER SYMMETRY WITH ELECTRICALLY CHARGED QUARKS

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Confinement

pure glue

Center symmetry
Center vortices

Interfaces in spin models

SU(N) in 2+1 d ↔ 2d Ising, Potts models

unquench...

fate of center symmetry?
→ a form persists when elec. charge of quarks included

relevance to phase diagram of Standard Model?

toy model – SU(2)xU(1)/Z₂
PART I

pure glue
**Center Symmetry**

- Symmetry of pure SU($N$) gauge theory

$$SU(N)/\mathbb{Z}_N, \quad z = e^{in\pi/N} \in \mathbb{Z}_N$$

- Center transformation

- Action invariant

- Polyakov loop picks up phase

$$P = \text{Tr} \uparrow \rightarrow z \text{ Tr} \uparrow$$

(time dir.)
**Center Symmetry — Confinement/Deconfinement**

- VEV determines whether center symmetry is realized

- Low $T$, confined, *center symmetric* phase

  \[
  \langle P \rangle = 0 \\
  \text{average to zero, disordered}
  \]

- High $T$, deconfined, *center broken* phase

  \[
  \langle P \rangle \neq 0 \\
  \text{sector spontaneously chosen}
  \]
**Vortices and Confinement**

- **Center vortex** – line/sheet in 3d/4d with flux quantized by center $Z_N$

- **Spacelike vortices** link with timelike Wilson loops
  - pick up phase due to flux

- **Low $T$ - spacelike vortices disorder**
  - area law for Wilson loop
  - vanishing Polyakov loop
  - confinement

- **High $T$ - spacelike vortices suppressed**

  (squeezed in temporal direction)


**TWIST AND CENTER VORTICES**

- ‘t Hooft’s twisted b.c.’s for pure SU(N)
  - fix the number (mod N) of center vortices through each plane
  - one non-periodic link per slice

\[ U \rightarrow zU \]

- flipped’ plaquettes

\[ \begin{array}{c}
\text{flux} \\
\hline
\end{array} \quad \rightarrow \quad z \begin{array}{c}
\text{flux} \\
\hline
\end{array} \]

\[ z = \exp(i2\pi n/N) \]
Ensembles with **color-electric flux** are obtained from the twisted ones

- Polyakov loop picks up twist
  \[ P(x + L_1) = z P(x) \]

- Correlators
  \[ \langle \uparrow \downarrow \rangle_{\text{twist}} = z \]

- Ensemble average over twist sectors gives **electric flux** partition function
  \[ Z_e(\vec{e}) = e^{-F_e(\vec{e})/T} \]

\( Z_N \) Fourier transform
CONFINEMENT PICTURE

\[ Z_{tw} = e^{-F_{cv}/T} \]

\[ Z_{el} = e^{-F_{el}/T} \]

Ph. De Forcrand, L. von Smekal
PRD 66 (2002) 011504 (R)
SVETITSKY-YAFFE CONJECTURE

(d+1)-dim pure gauge theory with 2nd order deconfinement transition, center $Z_N$

universality

d-dim $Z_N$ – spin model

- So...
  - 3+1 d SU(2) - 3d Ising model
  - 2+1 d SU(2) - 2d Ising model
  - 2+1 d SU(3) - 3-state Potts model

$\approx$ same theory near $T_c$
**Boundary Conditions**

- **Universality**: Polyakov loops $\leftrightarrow$ spins
  - Polyakov loop b.c. picks up twist element $z$

  $$U_t(x + L_1) = zU_t(x) \rightarrow P(x + L_1) = zP(x)$$

- Spin model: spins non-periodic by $z$
  $$s(x + L_1) = zs(x)$$
SO... CENTER VORTICES – SPIN INTERFACES

- e.g.
  - SU(2) in 2+1 d corresponds to 2d Ising
  - SU(2,3) in 2+1 d correspond to 2d Ising and 3-state Potts model
  - Many exact results available!
  - Exploit!

SU(2) in 2+1 d     2d Ising (N x N)

\[
z = -1
\]
SU(2) CRITICAL LATTICE COUPLINGS

- e.g. 
  SU(2) in 2+1 d
  2d Ising (N x N)

At $T_c$, for each $N_t$

$$\lim_{N_s \to \infty} \frac{Z_{tw}}{Z_0} = \lim_{N \to \infty} \frac{Z_{ap}}{Z_{pp}} = \frac{1}{1 + 2^{3/4}}$$

Use to precisely locate the critical lattice couplings

$$T_c \leftrightarrow \beta_c (N_t)$$
SU(2) CRITICAL LATTICE COUPLINGS

Intersect vortex free energies with Ising value for various spatial sizes $N_s$

$N_t = 5$

$$e^{-\frac{F_{cv}}{T}} = \frac{Z_{tw}}{Z_0}$$

$$= \frac{1}{1 + 2^{3/4}} + b(\beta - \beta_c)N_s^{1/\nu} + cN_s^{-\omega}$$

Extrapolate $\Rightarrow \beta_c (N_t)$
SU(2) CRITICAL COUPLINGS VS $N_T$

\[ \frac{\beta_c(N_t)}{4} = \frac{T_c}{g_3^2} N_t + c_1 + c_2 \frac{g_3^2}{T_c} \frac{1}{N_t} \]

from expansion of bare coupling in renorm’ed coupling
SU(2) CRITICAL COUPLINGS VS $N_T$

\[ \frac{T_c}{g_3^2} = 0.3757(5) \rightarrow \frac{T_c}{\sqrt{\sigma}} = 1.1225(23) \]

c.f. lit. 0.385(10) using $\sqrt{\sigma}/g_3^2$ from Liddle, Teper, arXiv:0803:2128

\[ \frac{\beta_c(N_t)}{4} = \frac{T_c}{g_3^2} N_t + c_1 + c_2 \frac{g_3^2}{T_c} \frac{1}{N_t} \]

from expansion of bare coupling in renorm’ed coupling
FINITE SIZE SCALING

Plot in terms of FSS variable

\[ x = T_c L t \propto \frac{L}{\xi_{\pm}} \]

\[ L = N_s a, \ t = (T/T_c - 1), \ \xi = \text{corr. length} \]

data for diff. \( N_s \) collapse
FINITE SIZE SCALING

Plot in terms of FSS variable

\[ x = T_c L t \propto \frac{L}{\xi} \]

\[ L = N_s a, \ t = (T/T_c - 1), \ \xi = \text{corr. length} \]

data for diff. \( N_s \) collapse

from \( Z_2 \) Fourier transform of twisted partition functions (all combinations of spacelike center vortices)
Finite size scaling

mirror $\rightarrow$ self duality for SU(2)!

$Z_{tw}/Z_0, Z_{e=1}/Z_{e=0}$ vs $x = T_c Lt$
Reflects **self-duality** of 2d Ising model *in a finite volume*

- \( Z_2 \) Fourier transform over ± b.c.'s \( \equiv \) Kramers-Wannier **duality** transform
- \( N \times N \) lattice

\[ \tilde{\beta} = -\frac{1}{2} \ln \tanh \beta \]

Savit, Rev. Mod. Phys. 52 (1980) 453
\( Z_2 \) Fourier transform over ± b.c.'s \( \equiv \) Kramers-Wannier duality transform

\[
\begin{pmatrix}
Z_{pp}(\tilde{\beta}) \\
Z_{ap}(\tilde{\beta}) \\
Z_{pa}(\tilde{\beta}) \\
Z_{aa}(\tilde{\beta})
\end{pmatrix} = \frac{1}{2} (\sinh 2\beta)^{-N^2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
Z_{pp}(\beta) \\
Z_{ap}(\beta) \\
Z_{pa}(\beta) \\
Z_{aa}(\beta)
\end{pmatrix}
\]

\[
\tilde{\beta} = -\frac{1}{2} \ln \tanh \beta, \quad \beta = \frac{J}{kT}
\]

Bugrij, Shadura hep-th/9601106

\( \tilde{\beta} - \beta_c = -(\beta - \beta_c) + \cdots \) \( \rightarrow \) FSS mirror
FSS fitting

- fit SU(2) to exact 2d Ising scaling function (1 parameter)

\[ x_I = Nt \approx \lambda_{Nt} x_{SU(2)} \]

- rescaling of correlation length since

\[ x \propto 1/\xi_{\pm} \]

- \( N_t \) dependent
  - need cont. limit
\[ \lambda(N_t) = \lambda(\infty) + \frac{b}{N_t} + \frac{c}{N_t^2} \]
FSS – DIFFERENT $N_t$ LATTICES
FSS — DIFFERENT $N_T$ LATTICES

Rescale:

$$x \rightarrow \lambda(N_t)x$$
STRING AMPLITUDES

Near $T_c$, tension of vortex and electric strings scale with $t = 1 - T/T_c$, e.g.

$$F_{tw}/T \to \tilde{\sigma}L \propto t, L \to \infty$$

- This is the large $x$ limit

FSS breaks down here

extract from exact Ising interface tension
2d Ising scaling function known exactly!

Asymptotically:

$$- \ln(\frac{Z_{ap}}{Z_{pp}}) \rightarrow \sigma_I N = 2 \ln(1 + \sqrt{2}) t N$$
2d Ising **scaling function** known **exactly**!

Fit SU(2) at small $x$ to obtain information at large $x$
STRING AMPLITUDES

- 2d Ising model \( F_I \rightarrow \sigma_I N = 2 \ln(1 + \sqrt{2})tN \)

- So SU(2) vortex/dual tension \( \tilde{\sigma} = \lambda T_c 2 \ln(1 + \sqrt{2})t, \, T > T_c \)

- And by self-duality

\[
\frac{F_e}{T} \rightarrow \sigma L / T
\]

\[
\sigma = \lambda T_c^2 2 \ln(1 + \sqrt{2})|t|, \quad \text{for } T < T_c
\]
SELF DUALITY FOR CRITICAL COUPLING

- 2d Ising model - intersecting $Z_{ap} / Z_{pp}$ and its dual from $Z_2$
  Fourier transform gives critical coupling exactly
  - self-dual in finite volume!

- SU(2) in 2+1 d - intersect $Z_{tw} / Z_0$ and $Z_e / Z_0$ for finite lattice

- Rapid convergence to infinite volume value!
SU(2) CONVERGENCE COMPARISON

$N_t = 4$

\[\beta_c(N_t, N_s)\]

\[Z_k - \text{uni}\]
\[Z_k - Z_e\]
SU(3) in 2+1 D

- universality class of 3-state Potts model
  - 3-state Potts model also self-dual in a finite volume!
  - again reflected in the gauge theory

![Graph showing self-duality in SU(3) (N_t = 2, N_s = 24)]
SU(3) – CRITICAL LATTICE COUPLINGS

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$\beta_c$</th>
<th>Lit.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8.15309(11)</td>
<td>8.1489(31)</td>
</tr>
<tr>
<td>4</td>
<td>14.7262(9)</td>
<td>14.717(17)</td>
</tr>
<tr>
<td>6</td>
<td>21.357(25)</td>
<td>21.34(4)</td>
</tr>
<tr>
<td>8</td>
<td>27.84(12)</td>
<td>-</td>
</tr>
</tbody>
</table>

\[
\frac{T_c}{g_3^2} = 0.5475(3) \rightarrow \frac{T_c}{\sqrt{\sigma}} = 0.9938(9)
\]

using \(\sqrt{\sigma}/g_3^2\) from \(^1\) c.f. lit. 0.9990(31) \(^1\)

SU(4) in 2+1 D

- Evidence for a 1st order transition

- Check with our twisted systems
  - violation of self-duality?
  - miss ‘universal’ value for large $N_s$?
  - FSS for slope of $Z_{tw}/Z_0$ not having e.g. Potts exponents?

- Not yet clear...
OTHER TRICKS

- Magnetic fluxes – spatial string tension

- ‘t Hooft Polyakov monopoles
  - SU(N) + Higgs GUT theories
  - C* with magnetic twist in 3+1 d to force monopoles on the lattice

- Electric fluxes at zero temp – string formation

**Summary – Part I**

- Confinement for pure SU(N) – center symmetry, vortices
- **Universality** - interfaces in spin models
- Exact results from 2d spin models – SU(2), SU(3) in 2+1 d
  - critical couplings, **self-duality**
unquench...
**Dynamical Quarks (Wilson)**

- explicitly break center symmetry – quarks see $Z_N$
- no more twist...

- Hopping expansion

$$\text{det } M = \exp\left(- \sum_j \frac{\kappa_j}{j} \text{Tr} H^j \right), \quad \kappa = \frac{1}{2am + 8}$$

$(1 - \gamma_\mu) U_\mu$

closed loops
Dynamical Quarks (Wilson)

- explicitly break center symmetry – quarks see $Z_N$
- no more twist...

Hopping expansion

$$\det M = \exp(-\sum_j \frac{\kappa_j}{j} \text{Tr} H^j), \quad \kappa = \frac{1}{2am + 8}$$

$$\ (1 - \gamma_\mu) U_\mu$$

closed loops

time dir.
DYNAMICAL QUARKS – $N_T = 4$

\[ S_{\text{eff}} = -\frac{\tilde{\beta}}{N_c} \sum \text{Re} \text{ Tr } \Box - 32\kappa^4 \sum_{\vec{x}} 2N_c \text{Re } P(\vec{x}) + \ldots \]

modify gauge coupling

ordering effect ($S_{\text{eff}}$ minimized for $P=1$)

c.f. spin system in magnetic field

Effect of fermions - ordering external field
Dynamical fermions – SU(2)

\[ P_{av} = \frac{1}{V} \sum_{\vec{x}} P \]

8^3 \times 4, \ \kappa = 0.15
...BUT QUARKS HAVE ELECTRIC CHARGE

- What if we include electromagnetism?

\[ q_u = +\frac{2}{3}e, \quad q_d = -\frac{1}{3}e \]

- Exactly compensate color center phase by U(1) phase

- **Gauge group**

\[ SU(3) \times U(1)_{em}/\mathbb{Z}_3 \]
**Hidden Symmetry**

- True Standard Model symmetry group

\[ SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6 \]

- Importance
  - unification, e.g. SU(5), SO(10) GUT
  - topological objects - color-EM monopoles/vortices

- Existence of a **global gauge symmetry** that may be spontaneously broken

  - relevant to **confinement**?

before electroweak trans. Zubhov, Veselov, Bakker
**Toy Lattice Model**

- 2 flavors of dynamical Wilson fermions, gauge group $SU(2) \times U(1)_{em}/\mathbb{Z}_2$

  - u/d quarks with $\pm \frac{1}{2}$ charge relative to $U(1)_{em}$ gauge action

  $$S = -\sum_{\Box} \left( \frac{\beta_{col}}{2} \text{Re} \ Tr \Box_{SU(2)} + \beta_{em} \cos \Box \theta \right) + S_{f,W}$$

- Implement via HMC

  $$U_\mu \exp \frac{i}{2} \theta$$

  Parallel transporters give both color and electromagnetic contribution to quarks – no net phase for $-1 \times -1 = 1$
**Effect of the U(1)**

- Expect $U(1)_{\text{em}}$ to have a **disordering effect**

- Recall Polyakov loop term from Hopping expansion

  \[ \propto -\text{Re} \text{ Tr} \uparrow \cdot \text{Re} \downarrow \]

  - color
  - EM

- If $U(1)$ loop random over space $\rightarrow$ c.f. spin model in a **random external field**

  \[ H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i h_i s_i \]
A WORD ON (PURE) COMPACT QED

- Confining at small values of the lattice coupling (i.e. strong coupling)
  - i.e. disordered phase for EM Polyakov loop

- Phase transition to Coulomb phase at $\beta \approx 1.01$

- Expect $U(1)$ to have large effect for small lattice coupling
RESTORATION OF QUENCHED BEHAVIOR FOR $P_{SU(2)}$

$SU(2) \times U(1)/\mathbb{Z}_2$

\[
8^3 \times 4, \; \kappa = 0.15 \; \text{(heavy quarks!)}
\]
IMPORTANCE OF QUARKS’ FRACTIONAL CHARGE

- U(1) angles in gauge action are **twice** those seen by quarks
  - phases differing by $\pi$ for quarks are not distinguished by the U(1) action

- As we cross $\beta_{em} \approx 1$, particles of unit electric charge should be deconfined - i.e. Polyakov loop of ‘electron’ becomes finite

- **BUT** there is still room for $Z_2$ disorder in the links as seen by quarks

Still disorder SU(2) Polyakov loop?

for quarks

for electrons / gauge action
HOT START - $P_{SU(2)}$

$8^3 \times 4$, $\kappa = 0.15$, $\sim 6000$ traj.
$U(1)$ gauge action is not able to remove the disorder for quarks, even deep in the Coulomb phase for unit charges.

$8^3 \times 4$, $\kappa = 0.15$, $\sim 6000$ traj.
Much to be understood...

- Cold starts – disorder only persists for a window beyond $\beta_{\text{em}} \approx 1$ – algorithmic issue only?

- Suppression of -1 transporters for light quarks? Competition between plaquette like terms

- **Speculation** for SU(3)xU(1) / $Z_3$
  
  move first order line towards lighter quarks?

  sharpen crossover if it doesn’t reach physical quark masses?
SUMMARY

- **Center symmetry** recovered when $U(1)$ is added to QCD with dynamical quarks

- **Disordering** effect of $U(1)$
  - How much can electromagnetism influence color dynamics?

- First steps in a toy $SU(2) \times U(1)/Z_2$ model
OUTLOOK

- Production runs – respectable masses and volumes
- Related spin models – random field Ising/Potts
- Hopping expansion – baby simulations

- Twisted boundary conditions!
  - in presence of dynamical fermions
  - combined vortices carrying both color and EM flux
COLD START - $P_{SU(2)}$

8^3 \times 4, \kappa = 0.15, \sim 40000 \text{ traj.}
COLD START - $P_{\text{SU}(2)}$

$8^3 \times 4$, $\kappa = 0.15$, $\sim 40000$ traj.
COLD START - $P_{\text{SU}(2)}$

All links start at 1 – for large $\beta_{\text{em}}$ it’s very tough for algorithm to reach -1 links!

$8^3 \times 4$, $\kappa = 0.15$, ~ 40000 traj.
CHECKING U(1) POLYAKOV LOOPS — COLD START

\[ \langle |P_{\text{aw}}| \rangle \ versus \ \beta_{\text{col}} \]

- \( P_{\text{quark}} \), \( \beta_{\text{col}} = 2.5 \)
- \( P_{\text{elec}} \)

jump
\[ \propto -\kappa^4 \, \text{Re} \, \text{Tr} \, \quad \text{Re} \, \theta/2 \]

**color**    **EM**

**PLAQ-PLAQ COMPETITION**
SU(3) CORR. CRITICAL EXPONENT

- FSS implies

\[ F_{tw} = \text{const.} + dN_s^{1/\nu} (\beta - \beta_c) + \cdots \]

- extract \( \nu \) from slopes of vortex free energy at crit.

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.812(17)</td>
</tr>
<tr>
<td>4</td>
<td>0.80(6)</td>
</tr>
<tr>
<td>Potts</td>
<td>( 5/6 = 0.833\ldots )</td>
</tr>
</tbody>
</table>

![Graph showing critical exponent from FSS](image)